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Lipschitz continuity of the spectra of the magnetic transition operators on a crystal lattice

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Dedicated to Professor Koichi Ogiue on his 60th birthday

Abstract

A magnetic transition operator on a crystal lattice is defined as a generalization of the Harper operator. Following the idea of J. Bellissard, we prove Lipschitz continuity of the band edges of its spectrum as magnetic field changes.
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1. Introduction

The Harper operator was designed to describe the behavior of an electron moving on the square lattice exposed to the constant magnetic field by Harper [6]. While the spectrum of the magnetic Laplacian of $\mathbb{R}^2$ under the uniform magnetic field is very simple and completely understood as the Landau levels, the spectrum of the Harper operator is difficult to analyze. The spectrum is a band when the magnetic flux is a rational number and is a Cantor set when the magnetic flux class is a Liouville number. Thus it has caught people's interest and much work has been done (cf. [3,5,7] and references therein).

As a generalization of the classical Harper operator on the square lattice, the notion of *magnetic transition operators* on more general graphs, namely crystal lattices, was introduced by Sunada [11]. The purpose of the present paper is to study the spectrum of the

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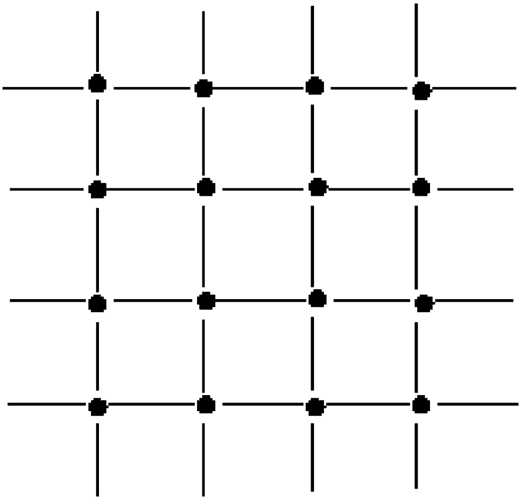

Fig. 1. The square lattice $\mathbb{Z}^2$.

magnetic transition operators on crystal lattices, especially how the spectrum depends on the associated magnetic field by using the C^*-algebra approach following Bellissard [2,3].

To state our main result, let us recall the definition of *magnetic transition operators*. A *crystal lattice* X is an infinite graph on which an abelian group Γ acts freely with a finite graph X_0 as its quotient, or equivalently, it is the abelian covering graph of a finite graph X_0 with the covering transformation group Γ. Intuitively, it is an infinite graph with a fundamental pattern consisting of finite vertices and finite edges, which appears periodically. The square lattice $\mathbb{Z}^d$, the triangular lattice, the hexagonal lattice are typical examples (see Figs. 1–3). For simplicity, we assume Γ has no torsion, therefore $\Gamma \cong \mathbb{Z}^d$ for some d.

Recall that a magnetic Laplacian of $\mathbb{R}^d$ under a periodic magnetic field B with respect to a lattice Γ is defined by a vector potential A, which is a *weakly Γ-invariant* 1-form of $\mathbb{R}^d$ satisfying $\mathrm{d}A = B$. (Here we consider B as a closed 2-form on $\mathbb{R}^d$.) A discrete analogue of a vector potential for a crystal lattice X is, therefore, a weakly Γ-invariant 1-form ω on

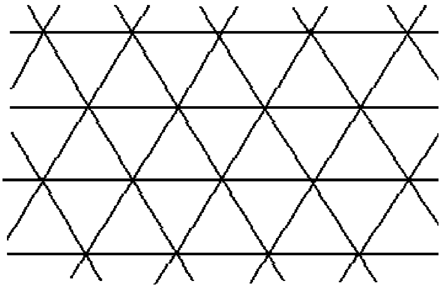

Fig. 2. The triangular lattice.

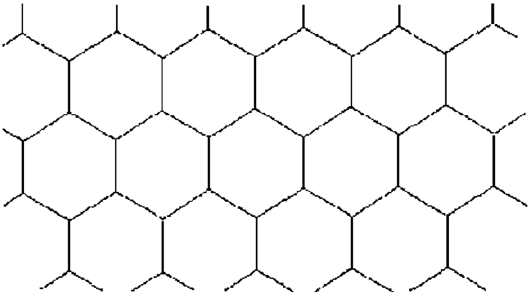

Fig. 3. The hexagonal lattice.

X (defined in Section 2). It defines a Γ-invariant cohomology class $[\omega] \in H^1(X, \mathbb{R})^\Gamma$. The covering map $\pi : X \to X_0$ induces the surjective map $\Theta : H^1(X, \mathbb{R})^\Gamma \to H^2(\Gamma, \mathbb{R})$, where $H^2(\Gamma, \mathbb{R})$ is the 2nd group cohomology of Γ. We call $\Theta[\omega]$ the *magnetic flux class* of $[\omega]$. For $\Gamma = \mathbb{Z}^d$, it is known that $H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}^{d(d-1)/2}$ and we identify $H^2(\Gamma, \mathbb{R})$ with the space of the skew symmetric bilinear form $B = \sum b_{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j$ of $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$, therefore it is a discrete analogue of the space of the magnetic flux classes of periodic magnetic fields on the Euclidean space.

For a weak Γ-invariant 1-form ω, the *magnetic transition operator* H_ω on X is defined by

$$(H_\omega \varphi)(x) = \sum_{e \in E_x} p(e)\, \mathrm{e}^{-\sqrt{-1}\omega(e)} \varphi(t(e)).$$

(See Section 2 for the definitions of the notations.) If two weak Γ-invariant 1-forms ω_1 and ω_2 represent the same element in $H^1(X, \mathbb{R})^\Gamma$(and so have the same magnetic flux class in $H^2(\Gamma, \mathbb{R})$), H_{ω_1} and H_{ω_2} are unitarily equivalent. Therefore, when spectrum is concerned, we say the magnetic transition operator $H_{[\omega]}$ corresponding to the magnetic flux class $B = \Theta[\omega]$. We shall see that H_ω is an element of the reduced twisted group C^*-algebra $\mathcal{A}_B$ associated with $B = \Theta[\omega] \in H^2(\Gamma, \mathbb{R})$ in Section 3.

An advantage of the C^*-algebra approach is that one can treat not only the magnetic transition operators but also a wider class of the operators which depends smoothly on the magnetic flux class. Let Ω be a small neighborhood of B_0 in $H^2(\Gamma, \mathbb{R})$ and $\mathcal{A}_\Omega = \cup_{B \in \Omega} \mathcal{A}_B$. Given a smooth structure on $\mathcal{A}_\Omega$, we define the space $\mathcal{C}^{l,n}(\mathcal{A}_\Omega)$ of a (l, n)-differentiable elements H in $\mathcal{A}_\Omega$ in Section 4.

Theorem 1.1. *Let $H \in \mathcal{C}^{1,d/2+2+\epsilon}(\mathcal{A}_\Omega)$ be a self-adjoint operator. Denote the upper/lower edges of a gap g of the spectrum of H by $E^g_\pm$, respectively. At $B_0 \in \Omega$ where the gap width W^g is positive, $E^g_\pm$ are Lipschitz continuous functions in B. Namely we have*

$$|E^g_\pm(B_2) - E^g_\pm(B_1)| \le c(H) \left[\sup_{B \in U(B_0)} W^g(B) \right]^{-(d/2+4)} |B_2 - B_1|,$$

for $B_1, B_2 \in U(B_0)$, where $U(B_0)$ is a small neighborhood of B_0 in *Ω in which W^g is positive.*

Remark 1.2. For arbitrary $\Gamma \cong \mathbb{Z}^d$, there are examples of crystal lattices such that the spectra of the magnetic transition operators on them with small magnetic flux classes have gaps. See Section 7.

Remark 1.3. When the magnetic flux class belongs to $H^2(\Gamma, \mathbb{Q})$, it is shown that the spectrum of the magnetic transition operator has a band structure in [8,11]. The Lipschitz constant of $\|H_\omega\|$ at the zero magnetic flux class is estimated in [8]. More precisely, for H_ω with the non-degenerate magnetic flux $B = \Theta[\omega]$,

$$\limsup_{\delta \to 0} \frac{1}{\delta^2}(1 - \|H_{\delta^2\omega}\|) \leq \frac{1}{m(X_0)} \sum |b_i|,$$

where $\pm\sqrt{-1}b_i$ are the eigenvalues of B.

Remark 1.4. The spectrum of the discrete magnetic Laplacian on Z^3-lattices (the $\mathbb{Z}^3$-cover of the 3-bouquet graph) is studied carefully by Bédos [1].

2. Magnetic transition operators

A magnetic field on $\mathbb{R}^d$ is a closed 2-form $B = \sum b_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$ and a vector potential A for B is a 1-form satisfying $\mathrm{d}A = B$. The magnetic Laplacian is the self-adjoint operator $\Delta_A = \nabla_A^* \nabla_A$, where $\nabla_A = d - \sqrt{-1}A$ is the connection of the trivial line bundle on $\mathbb{R}^d$ and ∇_A^* is its adjoint. The magnetic Laplacian associated with two different vector potentials for the same magnetic field B belong to the same unitary equivalence class of the operators. A magnetic field B is periodic with respect to a lattice group $\Gamma \subset \mathbb{R}^d$ if and only if $\gamma^* A - A = \mathrm{d}f_\gamma$ $(\gamma \in \Gamma)$. We call this property for A *weak Γ-invariant*.

In this section, we define the magnetic transition operator on a crystal lattice, as a discrete analogue of the magnetic Laplacian on $\mathbb{R}^d$. In the classical Harper model case, the square lattice lies in the Euclidean space $\mathbb{R}^2$ exposed to the perpendicular magnetic field (which is identified with a 2-form on $\mathbb{R}^2$). In our case, we consider the crystal lattice X as an abstract infinite graph. It is not in a Euclidean space a priori. Therefore, an account of what is the magnetic field/flux class corresponding to our magnetic transition operator is needed.

Let X be a crystal lattice, the covering graph of a finite graph X_0 with the covering transformation group $\Gamma \cong \mathbb{Z}^d$. Denote the space of all oriented edges of X by E, the origin and the terminus of an oriented edge e by $o(e)$, $t(e)$, the inverse edge of e by $\bar{e}$, respectively, and put $E_x = \{e \in E | o(e) = x\}$ for $x \in X$. Here and throughout the paper, we identify the space of vertices of X with X (so a vertex x is denoted as $x \in X$). Give a Γ-invariant *weight* $m : X \to \mathbb{R}_+$ of the vertices of X and a Γ-invariant transition probability p of a symmetric random walk on X, i.e. $p : E \to \mathbb{R}_+$ s.t.

$$\sum_{e \in E_x} p(e) = 1, \quad x \in X, \qquad m(o(e))p(e) = m(t(e))p(\bar{e}), \quad e \in E.$$

A *simple random walk*, i.e. $p(e) = \deg(o(e))^{-1}$, $m(x) = \deg(x)$ is an example of symmetric random walks on X.

As a discrete analogue of vector potentials, we take a *weakly Γ-invariant* 1-form ω on X, i.e. a function $\omega : E \to \mathbb{R}$ of E satisfying

$$\omega(\bar{e}) = -\omega(e), \qquad \gamma^*\omega - \omega = \mathrm{d}s_\gamma \quad (\forall\gamma \in \Gamma, \exists s_\gamma : X \to \mathbb{R}).$$

There is no straightforward discrete analogue of a magnetic field on X, because a magnetic field on the Euclidean space is a closed 2-form while X is a one-dimensional object. A weak Γ-invariant 1-form defines an element of $H^1(X, \mathbb{R})^\Gamma$, We take a Γ-invariant cohomology class $[\omega] \in H^1(X, \mathbb{R})^\Gamma$ as a substitute for a magnetic field.

For an element $[\omega] \in H^1(X, \mathbb{R})^\Gamma$, we define its *magnetic flux class* as an element of the 2nd group cohomology $H^2(\Gamma, \mathbb{R})$. A 2-*cocycle* B is a map $B : \Gamma \times \Gamma \to \mathbb{R}$ satisfying the cocycle condition:

$$B(\sigma_1, \sigma_2\sigma_3) + B(\sigma_2, \sigma_3) = B(\sigma_1, \sigma_2) + B(\sigma_1\sigma_2, \sigma_3) \quad (\sigma_1, \sigma_2, \sigma_3 \in \Gamma), \tag{1}$$

and the 2nd cohomology class is the equivalence class defined by the relation;

$$B_1 \sim B_2 \Leftrightarrow B_2(\alpha, \beta) = B_1(\alpha, \beta) + s(\beta) + s(\alpha) - s(\alpha\beta) \quad (\alpha, \beta \in \Gamma, \exists s : \Gamma \to \mathbb{R}).$$

From (1), we deduce

$$B(1, \sigma) = B(\sigma, 1) = B(1, 1),$$

and hence we always normalize B so that $B(1, \sigma) = B(\sigma, 1) = 0$.

For a given weak Γ-invariant ω ($\gamma^*\omega - \omega = \mathrm{d}s_\gamma$), put

$$B(\alpha, \beta) = s_\alpha(x) - s_{\alpha+\beta}(x) + s_\beta(\alpha^{-1}x) \quad (x \in X, \alpha, \beta \in \Gamma). \tag{2}$$

It does not depend on $x \in X$ and satisfies the cocycle condition. Moreover, the cohomology class does not depend on the choice of s_α, s_β, and ω (but on $[\omega]$). Thus it defines a map $\Theta : H^1(X, \mathbb{R})^\Gamma \to H^2(\Gamma, \mathbb{R})$. Actually there is an exact sequence;

$$0 \to H^1(\Gamma, \mathbb{R}) \xrightarrow{\iota} H^1(X_0, \mathbb{R}) \xrightarrow{\pi^*} H^1(X, \mathbb{R})^\Gamma \xrightarrow{\Theta} H^2(\Gamma, \mathbb{R}) \to 0.$$

When X is the universal abelian covering X_0^{ab} of X_0 (whose covering transformation group is $H_1(X_0)$), $\Theta : H^1(X, \mathbb{R})^\Gamma \to H^2(\Gamma, \mathbb{R})$ is isomorphism. A general abelian cover X of X_0 is a sub-cover of X_0^{ab} and Θ is surjective.

For $\Gamma \cong \mathbb{Z}^d$, it is known that $H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}^{d(d-1)/2}$ and we identify $H^2(\Gamma, \mathbb{R})$ with the space of the skew symmetric bilinear form $B = (b_{ij})$ of $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$, therefore with the space of the magnetic flux classes of periodic magnetic fields on $\Gamma \otimes \mathbb{R}$, and for later use, we also identify the space with that of the constant magnetic fields on $\Gamma \otimes \mathbb{R}$. We call $\Theta[\omega]$ the *magnetic flux class* of $[\omega]$.

Let

$$\ell^2(X) := \left\{ \varphi : X \to \mathbb{C} \,\middle|\, \|\varphi\|^2 = \sum_{x \in X} m(x)|\varphi(x)|^2 < \infty \right\}.$$

The *magnetic transition operator* $H_\omega : \ell^2(X) \to \ell^2(X)$ on X is defined for a weak Γ-invariant 1-form ω in [11] by

$$(H_\omega\varphi)(x) = \sum_{e\in E_x} p(e)\,\mathrm{e}^{-\sqrt{-1}\omega(e)}\varphi(t(e)).$$

Note that H_0 is the transition operator of the symmetric random walk on X. Just like the Euclidean case, two weak Γ-invariant ω_1, ω_2 yield unitarily equivalent magnetic transition operators and so it makes sense to say that the magnetic flux class $B = \Theta([\omega]) \in H^2(\Gamma, \mathbb{R})$ corresponds to H_ω.

We find, for an arbitrary element $B \in H^2(\Gamma, \mathbb{R})$, a canonical magnetic transition operator H_B on X whose corresponding magnetic flux class is B as follows. An arbitrary crystal lattice is realized in $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$ by the energy minimizing Γ-equivariant map $\Phi : X \to \Gamma \otimes \mathbb{R}$ and is called the *standard realization* [9]. It gives the most symmetric realization of the given crystal lattice. For example, the standard realization of the $\mathbb{Z}^2$-lattice is the square lattice, that of the triangular lattice is the equilateral triangular lattice, and that of the hexagonal lattice is the equilateral hexagonal lattice. The flat metric associated with the standard realization is called the *Albanese metric*. We identify B with the skew symmetric matrix $B = (b_{ij})$ by $B(\alpha, \beta) = \langle B\alpha, \beta\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner-product on $\Gamma \otimes \mathbb{R}$ defined by the Albanese metric. Consider the constant magnetic field $B = \sum_{i,j} b_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$ of $\Gamma \otimes \mathbb{R}$ and take a linear vector potential $A = \sum a_{ij}x_j\,\mathrm{d}x_i$ with $a_{ji} - a_{ij} = 2b_{ij}$, $a_{ij} \in \mathbb{R}$ associated with B. Put $\mathbf{a} = (a_{ij})$. Then

$$\omega_A(e) := \int_e \Phi^*A = \langle \mathbf{a}\Phi(o(e)), \mathrm{d}\Phi(e)\rangle + \frac{1}{2}\langle \mathbf{a}\mathrm{d}\Phi(e), \mathrm{d}\Phi(e)\rangle$$

is a weak Γ-invariant 1-form of X with $s_\gamma(x) = -\langle \mathbf{a}\gamma, \Phi(x)\rangle$ and its magnetic flux class is equal to B. Therefore we have H_{ω_A} whose corresponding magnetic flux class is the given B. We denote it by H_B hereafter.

On the other hand, when a weak Γ-invariant 1-form ω satisfies a certain condition (which should be considered to be the condition for ω to be a "linear vector potential" of X), there is the unique linear vector potential A of $\Gamma \otimes \mathbb{R}$ such that the ω is "essentially" the pull-back ω_A of A through the standard realization. In this case, we have a convergence theorem (CLT) of semigroups $H^n_{\omega/n} \to \mathrm{e}^{-t\Delta_A}$ [8], where Δ_A is the magnetic Laplacian with respect to the Albanese metric. Thus it is reasonable to call H_B the canonical magnetic transition operator for $B \in H^2(\Gamma, \mathbb{R})$.

Example 2.1 (the classical Harper operator). The square lattice is the $\mathbb{Z}^2$-cover of the 2-bouquet graph and is realized in $\mathbb{R}^2$ as the integer lattice. Denote the realization by $\Phi : \mathbb{Z}^2 \to \mathbb{R}^2$. Consider the vector potential $A = (1/2)(-by\,\mathrm{d}x + bx\,\mathrm{d}y)$ on $\mathbb{R}^2$ (b is a constant) whose corresponding constant magnetic field is $B = b\,\mathrm{d}x \wedge \mathrm{d}y$. The induced 1-form $\omega_A = \int \Phi^*A$ is a weak Γ-invariant 1-form on $\mathbb{Z}^2$, since $\gamma^*\omega - \omega = \mathrm{d}s_\gamma$ with $s_\gamma(x) = (1/2)B(\Phi(x), \gamma)$ and $\Theta([\omega]) = (1/2)B$. Give the transition probability for the simple random walk, i.e. $p(e) = 1/4$ for every $e \in E$. The magnetic transition operator H_ω

coincides with the classical Harper operator on $\mathbb{Z}^2$:

$$(H_\omega\varphi)(m,n) = \tfrac{1}{4}\Big[\mathrm{e}^{\sqrt{-1}bn/2}\varphi(m+1,n) + \mathrm{e}^{-\sqrt{-1}bn/2}\varphi(m-1,n) + \mathrm{e}^{-\sqrt{-1}bm/2}\varphi(m,n+1) + \mathrm{e}^{\sqrt{-1}bm/2}\varphi(m,n-1)\Big],$$

for $(m,n) \in \mathbb{Z}^2$.

3. Group C^* algebras

In the C^*-approach by Bellissard in [2,3], it is important to understand the Harper operator as an element of the non-commutative torus, the C^*-algebra consisting of the right magnetic translations which commutes with the left magnetic translations. Our magnetic transition operators are regarded as elements of a twisted C^*-algebra $\mathcal{A}(\Gamma, B, W)$ in the following way.

For a weak Γ-invariant ω, we take $s_\gamma(\gamma \in \Gamma)$ satisfying $\gamma^*\omega - \omega = \mathrm{d}s_\gamma$. The magnetic flux class $B \in H^2(\Gamma, \mathbb{R})$ is given as (2).

Let W be a finite dimensional Hilbert space and $\ell^2(\Gamma, W)$ be the space of ℓ^2 functions of Γ with W-valued. The *left magnetic translation* on $\ell^2(\Gamma, W)$ is defined by

$$(M_\alpha\phi)(\gamma) = \mathrm{e}^{-\sqrt{-1}B(\alpha,\alpha^{-1}\gamma)}\phi(\alpha^{-1}\gamma) \quad (\alpha, \gamma \in \Gamma, \phi \in \ell^2(\Gamma, W)),$$

and the *right magnetic translation* on $\ell^2(\Gamma, W)$ by

$$(U_\alpha\phi)(\gamma) = \mathrm{e}^{\sqrt{-1}B(\gamma,\alpha)}\phi(\gamma\alpha) \quad (\alpha, \gamma \in \Gamma, \phi \in \ell^2(\Gamma, W)).$$

It is straightforward to check

$$M_\alpha M_\beta = \mathrm{e}^{-\sqrt{-1}B(\alpha,\beta)}M_{\alpha\beta}, \qquad U_\alpha U_\beta = \mathrm{e}^{\sqrt{-1}B(\alpha,\beta)}U_{\alpha\beta}, \qquad M_\alpha U_\beta = U_\beta M_\alpha.$$

Put $\theta(\cdot,\cdot) = \mathrm{e}^{\sqrt{-1}B(\cdot,\cdot)}$ and

$$C(\Gamma, B, W) = \left\{A = \sum a_\alpha U_\alpha \ \text{ finite sum} | a_\alpha \in \mathrm{End}(W)\right\}.$$

Then $C(\Gamma, B, W)$ has the $*$-algebra structure by

$$\left(\sum a_\alpha U_\alpha\right)\left(\sum b_\beta U_\beta\right) = \sum \theta(\alpha,\beta)a_\alpha b_\beta U_{\alpha\beta},$$
$$*\left(\sum a_\alpha U_\alpha\right) = \sum a^*_{\alpha^{-1}}\theta(\alpha,\alpha^{-1})^{-1}U_\alpha.$$

The completion of $C(\Gamma, B, W)$ in $\mathcal{B}(\ell^2(\Gamma, W))$ with respect to the operator norm is denoted by $\mathcal{A}(\Gamma, B, W)$ and is called the *reduced twisted group C^*-algebra*. As Γ is an abelian group, it is isomorphic to the full twisted group C^*-algebra.

We shall relate H_ω with an element in $C(\Gamma, B, W) \subset \mathcal{A}(\Gamma, B, W)$ with $B = \Theta[\omega]$ and a suitable W. More generally, consider a self-adjoint operator $L : \ell^2(X) \to \ell^2(X)$ formally given, using the kernel function $h(\cdot,\cdot)$, by

$$(L\varphi)(x) = \sum_{y\in X} h(x,y)\varphi(y)$$

which commutes with all magnetic translations $\tilde{M}_\alpha$. Here by the magnetic translation, we mean

$$\tilde{M}_\alpha : \varphi(\cdot) \in \ell^2(X) \mapsto \mathrm{e}^{-\sqrt{-1}s_\alpha(x)}\varphi(\alpha^{-1}x) \in \ell^2(X) \quad (\varphi \in \ell^2(X), \alpha, \in \Gamma, x \in X).$$

Then the condition that $[L, \tilde{M}_\alpha] = 0$ is equivalent to the condition

$$h(x, y) = \mathrm{e}^{-\sqrt{-1}s_\alpha(x)} h(\alpha^{-1}x, \alpha^{-1}y)\, \mathrm{e}^{\sqrt{-1}s_\alpha(y)}. \tag{3}$$

We call this property *weak Γ-invariance*. The kernel function h of H_ω is given by

$$h(x, y) = \begin{cases} p(e)\,\mathrm{e}^{-\sqrt{-1}\omega(e)}, & x = o(e),\ y = t(e) \text{ with } \exists e \in E, \\ 0, & \text{otherwise}, \end{cases}$$

and is weak Γ-invariant.

Take a fundamental domain F of X for the Γ-action and put $W = \ell^2(F) \cong C(X_0)$, the $(\#X_0)$-dimensional vector space over $\mathbb{C}$. We shall see that a weak Γ-invariant operator L on $\ell^2(X)$ is regarded as an element of the von Neumann algebra:

$$\mathcal{W}(\Gamma, B, W) = \{A \in \mathcal{B}(\ell^2(\Gamma, W)) | [A, M_\alpha] = 0, \forall \alpha \in \Gamma\}.$$

We identify $\ell^2(X)$ with $\ell^2(\Gamma, W)$ by the correspondence:

$$\varphi \in \ell^2(X) \leftrightarrow \phi \in \ell^2(\Gamma, W),$$
$$\varphi(x) = \mathrm{e}^{-\sqrt{-1}s_\alpha(x)}\phi(\alpha)(x_0) \quad (\forall x = \alpha x_0, \exists! \alpha \in \Gamma, \exists! x_0 \in F).$$

Through this identification, the correspondent operator $\hat{L} : \ell^2(\Gamma, W) \to \ell^2(\Gamma, W)$ to $L : \ell^2(X) \to \ell^2(X)$ is given by

$$(\hat{L}\phi)(\alpha)(x_0) = \sum_{\gamma \in \Gamma} \theta(\alpha, \gamma) \sum_{y_0 \in X_0} \mathrm{e}^{-\sqrt{-1}s_\gamma(x_0)} h(\gamma^{-1}x_0, y_0)\phi(\alpha\gamma)(y_0).$$

As $\tilde{M}_\alpha$ corresponds to M_α under this identification, $\hat{L}$ belongs to $\mathcal{W}(\Gamma, B, W)$ if and only if L is weakly Γ-invariant.

By putting

$$(a_\gamma\psi)(x_0) = \sum_{y_0 \in X_0} \mathrm{e}^{-\sqrt{-1}s_\gamma(x_0)} h(\gamma^{-1}x_0, y_0)\psi(y_0),$$

formally we have $\hat{L} = \sum_{\gamma \in \Gamma} a_\gamma U_\gamma$.

In the case of the magnetic transition operator H_ω, $a_\gamma \neq 0$ if and only if there exits $e \in E$ with $o(e) = x_0 \in F$ and $t(e) = \gamma y_0$, $\exists x_0, \exists y_0 \in F$. For each $x_0 \in F$, $\#E_{x_0} < \infty$, and, for each $e \in E_{x_0}$, there is a unique γ satisfying $t(e) = \gamma y_0$. Thus there are only finite a_γ which do not vanish. Thus H_ω corresponds to an element of $C(\Gamma, B, W) \subset \mathcal{A}(\Gamma, B, W)$ with $W = C(X_0)$. From now on we identify H_ω with the corresponding element in $\mathcal{A}(\Gamma, B, W)$.

4. Smooth structure of the field of C^*-algebra

We put the Albanese metric on $\Gamma \otimes \mathbb{R}$ and identify $H^2(\Gamma, \mathbb{R})$ with the space of skew symmetric bilinear forms $B = (b_{ij})$ with respect to this flat metric. We always take the skew symmetric bilinear form as a representative of an element of $H^2(\Gamma, \mathbb{R})$, unless we mention otherwise.

For a given skew symmetric bilinear form $B \in H^2(\Gamma, \mathbb{R})$, as wee see in Section 2, there is the canonical weak Γ-invariant 1-form ω with $\Theta[\omega] = B$, and the magnetic transition operator $H_B = H_\omega$. We ask how $\mathrm{Spec}(H_B)$ depends on B when B changes smoothly. Since each H_ω belongs to a distinct $\mathcal{A}(\Gamma, B, W)$, we do not look at an individual $\mathcal{A}(\Gamma, B, W)$ but a field of C^*-algebras $\mathcal{A}(\Gamma, B, W)$'s. In this way, we treat not only H_ω but also a large class of smooth elements in Bellissard's formulation [2].

Take a small open subset $\Omega \subset H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}^{d(d-1)/2}$. Put

$$\mathcal{P}^k_\Omega = \left\{ A = \sum a_\alpha U^\Omega_\alpha \text{ finite sum} | a_\alpha \in C^k(\Omega, \mathrm{End}(W)) \right\},$$

where U^Ω_α's are formal unitary elements and $W = C(X_0)$. $\mathcal{P}^k_\Omega$ is equipped with $*$-algebra structure with the function $\theta : B \in \Omega \mapsto \theta(\cdot, \cdot) = \mathrm{e}^{\sqrt{-1}B(\cdot,\cdot)}$:

$$U^\Omega_\alpha U^\Omega_\beta = \theta(\alpha, \beta) U^\Omega_{\alpha\beta}, \qquad a_\alpha U^\Omega_\beta = U^\Omega_\beta a_\beta, \qquad (U^\Omega_\alpha)^* = (U^\Omega_\alpha)^{-1} = U^\Omega_{\alpha^{-1}}.$$

We also consider similarly

$$\mathcal{P}_{\{B\}} = \left\{ A = \sum a_\alpha U^B_\alpha \text{ finite sum} | a_\alpha \in \mathrm{End}(W) \right\},$$

and its completion $\mathcal{A}_B$ with respect to the C^*-norm

$$\|A\|_B := \sup_{\pi \in \mathrm{Rep}} \|\pi(A)\|,$$

where the supremum is taken over all unitary equivalence classes of representations of $\mathcal{P}_{\{B\}}$ on separable Hilbert spaces. Every representation of $\mathcal{P}_{\{B\}}$ or every $*$-automorphism of $\mathcal{P}_{\{B\}}$ extends uniquely to $\mathcal{A}_B$.

The C^*-algebra $\mathcal{A}(\Gamma, B, W)$ we have defined in the previous section is nothing but the right regular representation of $\mathcal{A}_B$ and isomorphic to $\mathcal{A}_B$, since Γ is abelian. Therefore our H_ω can be regarded as an element of $\mathcal{A}_B$ with $B = \Theta[\omega]$. Moreover, $B \in \Omega \to H_B$ belongs to $\mathcal{P}^\infty_\Omega$.

Define the evaluation homomorphism

$$\varrho_B : \mathcal{P}^k_\Omega \to \mathcal{P}_B \tag{4}$$

in the obvious way. The universal C^*-algebra $\mathcal{A}_\Omega$ is defined as the completion of P^k_Ω with respect to the norm

$$\|A\|_\Omega = \sup_{B \in \Omega} \|\varrho_B(A)\|_B.$$

The evaluation map extends to a $*$-homomorphism $\varrho_B : \mathcal{A}_\Omega \to \mathcal{A}_B$. The canonical trace $\tau : \mathcal{A}_\Omega \to C(\Omega)$ is given by $\tau(A)(B) = (\dim W)^{-1} \mathrm{tr}_W\, a_0(B) \in \mathbb{C}$. It is shown that $\mathcal{A}_\Omega$ is a continuous field of C^*-algebra in [4].

The field $\mathcal{A}_\Omega$ has the following smooth structure. We define a family of $*$-automorphism η_ζ of $\mathcal{A}_\Omega$ parameterized with $\zeta \in T^d = \Gamma \otimes \mathbb{R}/\Gamma$ by

$$\eta_\zeta(U_\alpha) = \mathrm{e}^{-2\pi\sqrt{-1}\langle\zeta,\alpha\rangle} U_\alpha. \tag{5}$$

For an element $A \in \mathcal{P}_\Omega^\infty$, we define

$$\partial_i A = \sum_{\alpha\in\Gamma} \sqrt{-1}\alpha_i a_\alpha U_\alpha \quad (i = 1, \dots, d), \qquad \delta_{ij} A = \sum_{\alpha\in\Gamma} \frac{\partial a_\alpha}{\partial b_{ij}} U_\alpha \quad (1 \le i < j \le d).$$

The ∂_i is a $*$-derivation and δ_{ij} satisfies

$$\delta_{ij}(*A) = *\delta_{ij}(A), \qquad \delta_{ij}(AB) = (\delta_{ij}A)B + A(\delta_{ij}B) - \sqrt{-1}(\partial_i A \partial_j B - \partial_j A \partial_i B).$$

Thus we define the order of ∂_i to be one and the order of δ_{ij} to be two. For example, differential operator $\delta^s\partial^r$ is of order $2|s| + |r|$ for multi-index s and r. The space

$$\mathcal{C}^{l,n}(\mathcal{A}_\Omega) = \{A \in \mathcal{A}_\Omega | \|\delta^s\partial^r(A)\| < \infty, 0 \le |s| \le l, 0 \le 2|s| + |r| \le n\}$$

is dense in $\mathcal{A}_\Omega$. It is shown in [2] that the $\mathcal{C}^{l,n}(\mathcal{A}_\Omega)$ has the norm $\|\cdot\|_{l,n}$ which makes $\mathcal{C}^{l,n}(\mathcal{A}_\Omega)$ a Banach $*$-algebra.

We also define the Sobolev space $\mathcal{H}^{l,n}$ as the completion of $\mathcal{P}_\Omega^\infty$ with respect to the norm

$$\begin{aligned}\|A\|_{\mathcal{H}^{l,n}} &:= \sup_{B\in\Omega} \sup\{\tau(|\delta^s\Delta^r A|^2)^{1/2} | 0 \le |s| \le l, 0 \le 2|s| + 2|r| \le n\} \\ &= \frac{1}{\sqrt{\dim W}} \sup_{B\in\Omega} \sup\{|\alpha|^{2r}(\mathrm{tr}_W|\delta^s a_\alpha|^2)^{1/2} | 0 \le |s| \le l, 0 \le 2|s| + 2|r| \le n\},\end{aligned}$$

where $\Delta = \sum \partial_i^2$. Note that $\|\cdot\|_{\mathcal{H}^{l,n}} \le C\|\cdot\|_{l,n}$. We extend the norms $\|\cdot\|_{\mathcal{H}^{l,n}}$ for real number $n \ge 2l > 0$, $l \in \mathbb{N} \cup \{0\}$.

5. Weyl representation

Let $W = C(X_0)$ and $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$ be the Euclidean space equipped with the Albanese metric throughout this section.

A representation of $\mathcal{A}_B$ on $L^2(\Gamma \otimes \mathbb{R}, W)$ is given by $U_\alpha^B \mapsto \tilde{U}_\alpha : L^2(\Gamma \otimes \mathbb{R}, W) \to L^2(\Gamma \otimes \mathbb{R}, W)$, a unitary operator:

$$(\tilde{U}_\alpha\varphi)(x) = \mathrm{e}^{\sqrt{-1}B(x,\alpha)}\varphi(x+\alpha) \quad (\varphi \in L^2(\Gamma \otimes \mathbb{R}, W)),$$

where we identify $\alpha \in \Gamma$ with the vector $\alpha \otimes 1 \in \Gamma \otimes \mathbb{R}$. These are the right magnetic translations on $L^2(\Gamma \otimes \mathbb{R}, W)$.

This representation has a direct integral decomposition by the right regular representations $\mathcal{A}(\Gamma, B, W) \subset \mathcal{B}(\ell^2(\Gamma, W))$ of $\mathcal{A}_B$. To be more precise, for $\varphi \in L^2(\Gamma \otimes \mathbb{R}, W)$ and $\zeta \in \Gamma \otimes \mathbb{R}/\Gamma$, we define $\iota_\zeta(\varphi) = \Psi_\zeta$ by

$$\Psi_\zeta : \gamma \in \Gamma \mapsto \mathrm{e}^{\sqrt{-1}B(\zeta,\gamma)}\varphi(\zeta+\gamma) \in W,$$

and associate a family $\Psi = \{\Psi_\zeta\}_{\zeta \in \Gamma\otimes\mathbb{R}/\Gamma}$ of $\ell^2_\xi(\Gamma, W)$ parameterized by $\zeta \in T^d = \Gamma\otimes\mathbb{R}/\Gamma$. Then it is easy to check

$$\|\Psi\|^2 := \int_{T^d} \|\Psi_\zeta\|^2_{\ell^2_\zeta(\Gamma,W)}\,\mathrm{d}\zeta = \int_{T^d} \sum_{\gamma\in\Gamma} \left\| \mathrm{e}^{\sqrt{-1}B(\zeta,\gamma)}\varphi(\zeta+\gamma)\right\|^2 d\zeta$$
$$= \int_{T^d} \sum_{\gamma\in\Gamma} \|\varphi(\zeta+\gamma)\|^2\,\mathrm{d}\zeta = \int_{\Gamma\otimes\mathbb{R}} |\varphi(x)|^2\,\mathrm{d}x = \|\varphi\|^2_{L^2(\Gamma\otimes\mathbb{R},W)}.$$

Thus $L^2(\Gamma\otimes\mathbb{R}, W) = \int^{\oplus}_{T^d} \ell^2_\zeta(\Gamma, W)\,\mathrm{d}\zeta$.

Let us see the induced operator of $\ell^2_\zeta(\Gamma, W) \cong \ell^2(\Gamma, W)$ coincides with the right magnetic translation $U_\alpha : \ell^2(\Gamma, W) \to \ell^2(\Gamma, W)$. Actually, we have

$$(\iota_\zeta(\tilde{U}_\alpha\varphi))(\gamma) = (\iota_\zeta(\mathrm{e}^{\sqrt{-1}B(\cdot,\alpha)}\varphi(\cdot+\alpha))(\gamma) = \mathrm{e}^{\sqrt{-1}B(\zeta,\gamma)}\,\mathrm{e}^{\sqrt{-1}B(\zeta+\gamma,\alpha)}\varphi(\zeta+\gamma+\alpha).$$

On the other hand, we get

$$U_\alpha(\iota_\zeta(\varphi))(\gamma) = \mathrm{e}^{\sqrt{-1}B(\gamma,\alpha)}\iota_\zeta(\varphi)(\gamma+\alpha) = \mathrm{e}^{\sqrt{-1}B(\gamma,\alpha)}\,\mathrm{e}^{\sqrt{-1}B(\zeta,\gamma+\alpha)}\varphi(\zeta+\gamma+\alpha).$$

Thus $\iota_\zeta\tilde{U}_\alpha = U_\alpha\iota_\zeta$.

Therefore the representation of $\mathcal{A}_B$ on $L^2(\Gamma\otimes\mathbb{R}, W)$ and that on $\ell^2(\Gamma, W)$ are the same and

$$\mathrm{Spec}(A : L^2(\Gamma\otimes\mathbb{R}, W)) = \bigcup_{\zeta\in T^d} \mathrm{Spec}(A : \ell^2_\zeta(\Gamma, W)) = \mathrm{Spec}(A : \ell^2(\Gamma, W)).$$

Instead of $\tilde{U}_\alpha$, we simply write $U_\alpha : L^2(\Gamma\otimes\mathbb{R}) \to L^2(\Gamma\otimes\mathbb{R})$ and extend the right magnetic translation U_α to U_v on $L^2(\Gamma\otimes\mathbb{R})$ for $v \in \Gamma\otimes\mathbb{R}$ and express it as an integral operator in the following way.

Let V be the orthogonal complement of the null space of B and its dimension $\dim(V) = 2k$. We write $x = x' + x'' \in V \oplus V^\perp$. Denote a slightly modified right translation on $L^2(V)$

$$(U_v\varphi)(x) = \mathrm{e}^{\sqrt{-1}B(x,v)}\varphi(x+v') \quad (x \in V \subset \Gamma\otimes\mathbb{R}, v \in \Gamma\otimes\mathbb{R})$$

by the same symbol. Note that $B(x, v) = B(x, v')$ as $v - v'$ belongs to the null space of B. It still satisfies the relation $U_vU_w = \theta(v, w)U_{v+w}$ and acts on $L^2(V, W) \subset L^2(\Gamma\otimes\mathbb{R}, W)$.

For a while, we work on $L^2(V, W)$. So, instead of writing x', etc. we use x, etc. Since ${}^tBB$ is a positive definite symmetric matrix on V, it defines the inner-product $\langle x, y\rangle_B = \langle x, |B|y\rangle_0$ and the volume form $\mathrm{d}_Bx = B\wedge\cdots\wedge B$ of V. Put $\varphi_0(x) = \pi^{-k/2}\,\mathrm{e}^{-12|x|^2_B}(\|\varphi\|_B = 1)$ and $L^2_0(V, W) = \mathrm{Span}\{U_{v'}\varphi_0\}_{v\in\Gamma\otimes\mathbb{R}} \subset L^2(V, W) \subset L^2(\Gamma\otimes\mathbb{R}, W)$.

Define $P : L^2(V, W) \to L^2(V, W)$ by

$$(P\varphi)(x) = \pi^{-k/2}\int_V \varphi(y)U_{-y}\varphi_0(x)\,\mathrm{d}_By.$$

By definition, the image $P(L^2(V, W))$ is contained in $L^2_0(V, W)$.

Lemma 5.1.

$$P\varphi(x) = \pi^{-k/2}\langle\langle \varphi, U_{-x}\varphi_0\rangle\rangle_B = \pi^{-k/2}\langle\langle U_x\varphi, \varphi_0\rangle\rangle_B, \quad PU_v = U_v P, \quad P\varphi_0 = \varphi_0.$$

Proof. First we observe that $U_{-y}\varphi_0(x) = \overline{U_{-x}\varphi_0(y)}$ and therefore

$$P\varphi(x) = \pi^{-k/2}\int_V \varphi(y)\overline{U_{-x}\varphi_0(y)}\,\mathrm{d}_B y = \pi^{-k/2}\langle\langle \varphi, U_{-x}\varphi_0\rangle\rangle_B.$$

Next by using

$$\langle\langle U_v\varphi, \psi\rangle\rangle_B = \langle\langle \varphi, U_{-v}\psi\rangle\rangle_B,$$

we have

$$P\varphi(x) = \pi^{-k/2}\langle\langle U_x\varphi, \varphi_0\rangle\rangle_B,$$

and also

$$\begin{aligned}(PU_v\varphi)(x) &= \pi^{-k/2}\langle\langle U_v\varphi, U_{-x}\varphi_0\rangle\rangle_B = \pi^{-k/2}\langle\langle \varphi, U_{-v}U_{-x}\varphi_0\rangle\rangle_B \\ &= \pi^{-k/2}\langle\langle \varphi, \theta(v,x)U_{-(v+x)}\varphi_0\rangle\rangle_B = \pi^{-k/2}\theta(x,v)\langle\langle \varphi, U_{-(x+v)}\varphi_0\rangle\rangle_B \\ &= \pi^{-k/2}\theta(x,v)\langle\langle U_{x+v}\varphi, \varphi_0\rangle\rangle_B = (U_v P\varphi)(x).\end{aligned}$$

We can check $P\varphi_0 = \varphi_0$ by direct calculation. □

Lemma 5.2. *P is the orthogonal projection onto $L_0^2(V, W)$.*

Proof. From the above lemma, for $\varphi \in L_0^2(V,W)^{\perp} \subset L^2(V,W)$, $P\varphi = 0$ and for every elements $\varphi = U_v\varphi_0$, we see that $P\varphi = PU_v\varphi_0 = U_v\varphi_0 = \varphi$. □

The kernel function of P is $p(x,y) = \pi^{-k/2}U_{-y}\varphi_0(x) = \overline{p(y,x)}$.

We define the *Weyl representation*, a projective representation of $\Gamma\otimes\mathbb{R}$, $\pi_w(U_v) = PU_{v'} : L_0^2(V,W) \to L_0^2(V,W)$ for $v \in \Gamma\otimes\mathbb{R}$, i.e.

$$(\pi_w(U_v)\varphi)(x) = \mathrm{e}^{\sqrt{-1}B(x,v)}\varphi(x+v').$$

It can be written as

$$\begin{aligned}(\pi_w(U_v)\varphi)(x) &= \pi^{-k/2}\langle\langle U_{v'}\varphi, U_{-x}\varphi_0\rangle\rangle_B = \pi^{-k/2}\langle\langle \varphi, U_{-v'}U_{-x}\varphi_0\rangle\rangle_B \\ &= \pi^{-k/2}\left\langle\!\!\left\langle \varphi, \mathrm{e}^{\sqrt{-1}B(v',x)}U_{-x-v'}\varphi_0\right\rangle\!\!\right\rangle_B \\ &= \pi^{-k/2}\,\mathrm{e}^{\sqrt{-1}B(x,v)}\int_V \mathrm{e}^{\sqrt{-1}B(y,x+v)}\varphi_0(y-x-v')\varphi(y)\,\mathrm{d}_B y.\end{aligned}$$

By the symplectic Fourier transform formula, we have the expression

$$\pi_w(U_v) = (4\pi)^{-k}\int_V \mathrm{e}^{\sqrt{-1}B(v,\xi)}\,\mathrm{e}^{(1/2)|v'|_B^2}T_\xi\,\mathrm{d}_B\xi, \tag{6}$$

where $T_\xi : L^2_0(V, W) \to L^2_0(V, W)$ is the integral operator with the kernel function

$$t_\xi(x, y) = \pi^{-k}\, \mathrm{e}^{(1/2)\sqrt{-1}B(\xi, y-x)}\, \mathrm{e}^{-(1/4)|y-x|^2_B}\, \mathrm{e}^{-(1/4)|y+x+\xi|^2_B}.$$

T_ξ is hermitian because $t_\xi(x, y) = \overline{t_\xi(y, x)}$. Putting $\varphi_\xi = U_{\xi/2}\varphi_0$, we see $T_\xi\varphi = \langle\langle\varphi, \varphi_\xi\rangle\rangle_B\varphi_\xi$, i.e. a one-dimensional projection. Putting $v = 0$ in (6),

$$(4\pi)^{-k}\int_V T_\xi\, \mathrm{d}_B\xi = \mathrm{Id}_{L^2_0(V,W)}. \tag{7}$$

By easy computation, we have

Lemma 5.3.

$$\mathrm{tr}(T_\xi T_\zeta) = \iint t_\xi(x, y)t_\zeta(y, x)\, \mathrm{d}_B x\, \mathrm{d}_B y = \mathrm{e}^{-(1/4)|\xi-\zeta|^2},$$

$$\mathrm{tr}(T_\xi) = \pi^{-k}\int \mathrm{e}^{-|2x+\xi|^2_B/4}\, \mathrm{d}_B x = 1.$$

We see in particular that $\Omega_\eta : A \mapsto \mathrm{tr}(T_\eta A)$ is a state of $\mathcal{B}(L^2_0(V, W))$.

6. Spectral gap

Now we want to compare the spectrum of H_{B_0} and H_{B_1} when B_0 and B_1 are close. For that let Ω be a small neighborhood of B_0 and write $B' = B_0 + hB \in \Omega$ with a small $h \in \mathbb{R}$. Let V be the orthogonal complement of the null space of B in $\Gamma\otimes\mathbb{R}$ and $x = x' + x'' \in V \otimes V^\perp$. Put rank $\Gamma = d$ and dim $V = 2k$. Consider the representation π_h of $\mathcal{A}_{hB}$ on $L^2_0(V, W)$ given by

$$(\pi_h(U^B_\alpha)\varphi)(x) = \left(\pi_w\left(U^B_{\sqrt{h}\alpha}\right)\varphi\right)(x) = \mathrm{e}^{\sqrt{-1}B(x,\sqrt{h}\alpha)}\varphi(x + \sqrt{h}\alpha') \quad (\alpha \in \Gamma),$$

where, in the last two terms, we identify $\alpha \in \Gamma$ with the vector $\alpha \otimes 1 \in \Gamma\otimes\mathbb{R}$ and consider $\sqrt{h}\alpha$, $\sqrt{h}\alpha'$ as vectors in $\Gamma\otimes\mathbb{R}$. It is easy to check

$$\pi_h(U^B_\alpha)\pi_h(U^B_\beta) = \mathrm{e}^{\sqrt{-1}hB(\alpha,\beta)}\pi_h(U^B_{\alpha\beta}) \quad (\alpha, \beta \in \Gamma).$$

We identify $\mathcal{A}_{B'}$ with the subalgebra $\mathcal{A}'$ generated by $\{U^{B_0}_\alpha \otimes \pi_h(U^B_\alpha)\}_{\alpha\in\Gamma}$ of $\mathcal{A}_B \otimes B(L^2_0(V, W))$ by the correspondence $U^{B'}_\alpha \leftrightarrow U^{B_0}_\alpha \otimes \pi_h(U^B_\alpha)$.

Proposition 6.1. *The $*$-homomorphism ι from $\mathcal{A}_{B'}$ to $\mathcal{A}'$ defined by $\iota : U^{B'}_\alpha \to U^{B_0}_\alpha \otimes \pi_h(U^B_\alpha)$ is an isomorphism.*

Proof. It is enough to show that ι is injective. Recall that the $*$-automorphism $\eta_\zeta : \mathcal{A}_{B_0} \to \mathcal{A}_{B_0}$ is given by

$$\eta_\zeta(U^{B_0}_\alpha) = \mathrm{e}^{-2\pi\sqrt{-1}\langle\zeta,\alpha\rangle}U^{B_0}_\alpha,$$

where $\zeta \in \Gamma \otimes \mathbb{R}/\Gamma$. By using it, we also define the $*$-automorphism

$$\tilde{\eta}_\zeta = \eta_\zeta \otimes 1 : \mathcal{A}_{B_0} \otimes \mathcal{B}(L_0^2(V, W)) \to \mathcal{A}_{B_0} \otimes B(L_0^2(V, W)).$$

By definition,

$$\tilde{\eta}_\zeta(U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B)) = \mathrm{e}^{-2\pi\sqrt{-1}\langle\zeta,\alpha\rangle} U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B),$$

and thus $\eta_\zeta \circ \iota = \iota \circ \tilde{\eta}_\zeta$, with $\eta_\zeta : \mathcal{A}_{B'} \to \mathcal{A}_{B'}$.

First, we show that ι is injective when it is restricted in $\mathcal{P}_{B'}$. Actually, for a finite sum $A = \sum a_\alpha U_\alpha^{B'}$ with $\iota(A) = 0$, we have

$$0 = \eta_\zeta(\iota(A)) = \iota(\tilde{\eta}(A)) = \sum a_\alpha \, \mathrm{e}^{-2\pi\sqrt{-1}\langle\zeta,\alpha\rangle} U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B).$$

By computing

$$\int_{T^d} \mathrm{e}^{2\pi\sqrt{-1}\langle\zeta,\gamma\rangle} \iota(\tilde{\eta}_\zeta(A)) \, \mathrm{d}\zeta = a_\gamma U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B),$$

we get $a_\gamma = 0$ for arbitrary $\gamma \in \Gamma$. That is $A = 0$ in $\mathcal{P}_{B'}$.

For a general $A \in \mathcal{A}_{B'}$ with $\iota(A) = 0$, we use an approximation argument. Take a series $\{f_n \in C(T^d)\}$ of trigonometric polynomials, i.e.

$$f_n = \sum_{|\gamma| \le N_n} c_{n,\gamma} \, \mathrm{e}^{2\pi\sqrt{-1}\langle\gamma,\zeta\rangle} \quad (c_{n,\gamma} = \text{const.}),$$

such that

1. $f_n \ge 0$,
2. $$\frac{1}{\mathrm{vol}(T^d)} \int_{T^d} f_n = 1,$$
3. for $\delta > 0$,
$$\lim_{n \to \infty} \int_{|\zeta| \ge \delta} f_n = 0.$$

A series of such $\{f_n\}$ is given by the Fejer polynomials (see [2]). Let $A \in \mathcal{A}_{B'}$ such that $\iota(A) = 0$ and put

$$A_n = \eta_{f_n}(A) := \int_{T^d} f_n(\zeta)\eta_\zeta(A) \, \mathrm{d}\zeta = \sum_\alpha \sum_{|\gamma| \le N_n} c_{n,\gamma} a_\alpha \int_{T^d} \mathrm{e}^{2\pi\sqrt{-1}\langle\gamma-\alpha,\zeta\rangle} U_\alpha^{B'}$$
$$= \sum_{|\gamma| \le N_n} c_{n,\gamma} a_\gamma U_\gamma^{B'} \in \mathcal{P}_{B'}.$$

Since $\iota \circ \eta_\zeta = \tilde{\eta}_\zeta \circ \iota$, we obtain

$$\iota(A_n) = \iota(\eta_{f_n}(A)) = \tilde{\eta}_{f_n} \iota(A) = \int f_n(\zeta) \tilde{\eta}_\zeta(\iota(A)) = 0.$$

As ι is injective on $\mathcal{P}_{B'}$, we conclude that $A_n = 0$.

On the other hand,

$$\begin{aligned}\|A - A_n\| &= \left\| \int_{T^d} f_n(\zeta)(A - \eta_\zeta(A)) \right\| \le \int_{T^d} f_n(\zeta)\|A - \eta_\zeta(A)\| \\ &\le \int_{|\zeta|<\delta} f_n(\zeta)\|A - \eta_\zeta(A)\| + 2\|A\| \int_{|\zeta|\ge\delta} f_n(\xi) \to 0,\end{aligned}$$

as $n \to \infty$. Therefore

$$\|A\| \le \lim_n \|A_n - A\| + \|A_n\| = 0,$$

i.e. $A = 0$. Thus we have proved ι is injective on $\mathcal{A}_{B'}$. □

Let $A \in \mathcal{H}_{l,n}(\mathcal{A}_\Omega)$ with $l \ge 0$ and $n > d/2 + 2$ such that

$$\int_{\Gamma\otimes\mathbb{R}/\Gamma} \mathrm{e}^{2\pi\sqrt{-1}\langle\zeta,\alpha\rangle} \eta_\zeta(A)\,\mathrm{d}\zeta = a_\alpha.$$

Thus A is formally written as $A = \sum a_\alpha U_\alpha^\Omega$. We write $\varrho_{B'}(A) = A(B')$, where $\varrho_{B'} : \mathcal{A}_\Omega \to \mathcal{A}_{B'}$ is the evaluation map (4). It can be written as

Lemma 6.2.

$$A(B') = (4\pi h)^{-k} \int_V A(h,\xi) \otimes T_{\xi/\sqrt{h}}\,\mathrm{d}_B\xi, \tag{8}$$

where

$$A(h,\xi) = \sum a_\alpha(B')\,\mathrm{e}^{\sqrt{-1}B(\alpha,\xi)}\,\mathrm{e}^{(h/2)|\alpha'|_B^2} U_\alpha^{B_0} \in \mathcal{A}_{B_0}.$$

Proof. Let

$$A(B') = \sum a_\alpha(B') U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B) \in \mathcal{A}_{B_0} \otimes \mathcal{B}(L_0^2(V,W)).$$

By using the expression (6), we have

$$\begin{aligned}A(B') &= (4\pi)^{-k} \int_V \sum a_\alpha(B') U_\alpha^{B_0} \otimes \mathrm{e}^{\sqrt{-1}B(\sqrt{h}\alpha,\xi)}\,\mathrm{e}^{(1/2)|\sqrt{h}\alpha'|_B^2} T_\xi\,\mathrm{d}_B\xi \\ &= (4\pi)^{-k} \int_V \sum a_\alpha(B')\,\mathrm{e}^{\sqrt{-1}B(\alpha,\sqrt{h}\xi)}\,\mathrm{e}^{(h/2)|\alpha'|_B^2} U_\alpha^{B_0} \otimes T_\xi\,\mathrm{d}_B\xi \\ &= (4\pi h)^{-k} \int_V \sum a_\alpha(B')\,\mathrm{e}^{\sqrt{-1}B(\alpha,\xi)}\,\mathrm{e}^{(h/2)|\alpha'|_B^2} U_\alpha^{B_0} \otimes T_{\xi/\sqrt{h}}\,\mathrm{d}_B\xi \\ &= (4\pi h)^{-k} \int_V A(h,\xi) \otimes T_{\xi/\sqrt{h}}\,\mathrm{d}_B\xi.\end{aligned}$$

□

By putting the $*$-automorphism $\eta_\xi^B : \mathcal{A}_{B_0} \to \mathcal{A}_{B_0}$ defined by $\eta_\xi(U_\alpha^{B_0}) = \mathrm{e}^{\sqrt{-1}B(\alpha,\xi)} U_\alpha^{B_0}$, we see $A(h,\xi) = \eta_\xi^B(A(h,0))$ and therefore $\mathrm{Spec}(A(h,\xi)) = \mathrm{Spec}(A(h,0))$ does not depend on ξ. Put $E(B') = \|A(B')\|$, $E(B_0) = \|A(B_0)\|$, and $E(h,\xi) = \|A(h,\xi)\| = E(h,0)$.

Lemma 6.3. *Let $A \in \mathcal{H}_{l,n}(\mathcal{A}_\Omega)$ for $l \geq 0$ and $n > d/2$, formally written as $A = \sum a_\alpha U_\alpha^\Omega$, and let $A_N = \sum_{|\alpha|<N} a_\alpha U_\alpha$. For $0 \leq \nu \leq n - d/2$,*

$$\|A - A_N\| \leq CN^{-\nu}\|A\|_{\mathcal{H}_{l,n}}.$$

Proof. Noting that U_α's are unitaries, i.e. $\|U_\alpha\| = 1$, we obtain

$$\|A - A_N\| = \left\| \sum_{|\alpha|\geq N} a_\alpha U_\alpha \right\| \leq \sum_{|\alpha|\geq N} |a_\alpha| \|U_\alpha\| = \sum_{|\alpha|\geq N} |a_\alpha|.$$

By Hölder inequality, we have

$$\sum_{|\alpha|\geq N} |a_\alpha| \leq \left(\sum_{|\alpha|\geq N} |\alpha|^{-4r} \right)^{1/2} \left(\sum_{|\alpha|\geq N} |\alpha|^{4r} |a_\alpha|^2 \right)^{1/2}.$$

If $4r - d = 2\nu > 0$, we have

$$\sum_{|\alpha|\geq N} |\alpha|^{-4r} \leq \int_{x\in\mathbb{R}^d, |x|\geq N} |x|^{-4r}\,\mathrm{d}x = C\nu^{-1}N^{-2\nu}.$$

On the other hand, as $n \geq 2r = \nu + d/2$ and $l \geq 0$,

$$\sum_{|\alpha|\geq N} |\alpha|^{4r} |a_\alpha|^2 \leq C\|A\|^2_{\mathcal{H}_{l,n}}.$$

Thus we have shown

$$\|A - A_N\| \leq CN^{-\nu}\|A\|_{\mathcal{H}_{l,n}},$$

for $0 \leq \nu \leq n - d/2$. □

Lemma 6.4. *Let $A \in \mathcal{H}_{l,n}(\mathcal{A}_\Omega)$ with $l \geq 1$ and $n > d/2 + 2$ and $0 < \nu \leq n - 2 - d/2$ and $A_N = \sum_{|\alpha|<N} a_\alpha U_\alpha^\Omega$, $A_N(B_0) = \varrho_{B_-}(A_N)$. Then we have*

$$\|A_N(h,0) - A_N(B_0)\| \leq Ch\|B\|N^{-\nu}\,\mathrm{e}^{(h\|B\|/2)N^2}\|A\|_{\mathcal{H}_{l,n}},$$

where $\|B\|^2 = \mathrm{tr}_W \mathcal{B}\mathcal{B}^* = \sum_{ij} b_{ij}^2$.

Proof. Since $U_\alpha^{B_0}$'s are unitary, we have

$$\begin{aligned}
&\|A_N(h,0) - A_N(B_0)\| \\
&\quad \le \sum_{|\alpha|<N} \|[a_\alpha(B')\,\mathrm{e}^{(h/2)|\alpha'|_B^2} - a_\alpha(B_0)]U_\alpha^{B_0}\| \\
&\quad \le \sum_{|\alpha|<N} \int_0^1 \left| \sum b_{ij} \frac{\delta a_\alpha}{\delta b_{ij}}(B_0 + sB) + \frac{|\alpha'|_B^2}{2} a_\alpha(B_0 + sB) \right| \mathrm{e}^{(h/2)|\alpha'|_B^2}\,\mathrm{d}s \\
&\quad \le h\|B\|\mathrm{e}^{(h\|B\|/2)N^2} \sup_{\Omega} \left(\sum_{|\alpha|<N} |\delta a_\alpha| + |\alpha|^2|a_\alpha| \right).
\end{aligned} \tag{9}$$

The first term of the R.H.S. of (9) is estimated as in Lemma 6.3. Namely, for $l \ge 1$ and $n \ge 2 + 2r$ with $4r - d = 2\nu$,

$$\sum_{|\alpha|<N} |\delta a_\alpha| \le CN^{-\nu}\|A\|_{\mathcal{H}_{l,n}}.$$

In our case, $2 + 2r = 2 + \nu + d/2 < n$ is fulfilled.

Now for the second term of the R.H.S. of (9), from the Hölder inequality, it follows that

$$\sum_{|\alpha|<N} |\alpha|^2|a_\alpha| \le \left(\sum_{|\alpha|<N} (|\alpha|^2 + 1)^{-2(r-1)} \right)^{1/2} \left(\sum_{|\alpha|<N} (|\alpha|^2 + 1)^{2r}|a_\alpha|^2 \right)^{1/2}.$$

We also get

$$\sum_{|\alpha|<N} (|\alpha|^2 + 1)^{-2(r-1)} \le \int_{x\in\mathbb{R}^d, |x|\le N} (1 + |x|^2)^{-2(r-1)} \le CN^{-2\nu},$$

with $2\nu = 4(r-1) - d > 0$. One the other hand, for $l \ge 0$ and $n \ge 2r$,

$$\sum_{|\alpha|<N} (|\alpha|^2 + 1)^{2r}|a_\alpha|^2 \le C\|A\|^2_{\mathcal{H}_{l,n}}.$$

Again, in our case, we have $2r = \nu + 2 + d/2 \le n$. Thus we have the assertion. □

From (7), it follows that

$$E(B') \le E(h,\xi)\|(4\pi h)^{-k} \int_V T_{\xi/\sqrt{h}}\,\mathrm{d}_B\xi\| = E(h,\xi). \tag{10}$$

Now we take $N = \mathrm{O}((h\|B\|)^{-1/2})$. Then from Lemmas 6.3 and 6.4 and (10), we have

$$\begin{aligned}
E(B') &\le E_N(B') + h\|B\|\|A\|_{\mathcal{H}_{l,n}} \le E_N(h,0) + h\|B\|\|A\|_{\mathcal{H}_{l,n}} \\
&\le E(B_0) + h\|B\|\|A\|_{\mathcal{H}_{l,n}}.
\end{aligned} \tag{11}$$

Lemma 6.5. *For $l \le 1$ and $n > 2 + d/2$,*

$$E(B') \ge E(B_0) - h\|A\|_{\mathcal{H}_{l,n}}.$$

Proof. For an arbitrary small $\epsilon > 0$. there is a state ω_ϵ of $\mathcal{A}_{B_0}$ such that $\omega_\epsilon(A(h,0)) \geq E(h) - \epsilon$. We also define the state $\Omega_\zeta(A) = \mathrm{tr}(T_{\zeta/\sqrt{h}}A)$ of $B(L_0^2(V,W))$.

$$\begin{aligned}
&(\omega_\epsilon \otimes \Omega_\eta)(A(B')) \\
&\quad = (4\pi h)^{-k} \int \omega_\epsilon(A(h,\xi))\mathrm{tr}(T_{\eta/\sqrt{h}}T_{\xi/\sqrt{h}})\,\mathrm{d}_B\xi \\
&\quad = (4\pi h)^{-k} \int \omega_\epsilon(A(h,\xi))\,\mathrm{e}^{-|\eta-\xi|_B^2/4h}\,\mathrm{d}_B\xi \\
&\quad = (4\pi h)^{-k} \sum a_\alpha(B')\,\mathrm{e}^{(h/2)|\alpha|_B^2}\omega_\epsilon(U_\alpha^{B_0}) \int \mathrm{e}^{\sqrt{-1}B(\alpha,\xi)}\mathrm{e}^{-|\eta-\xi|_B^2/4h}\,\mathrm{d}_B\xi \\
&\quad = \sum a_\alpha(B')\,\mathrm{e}^{-(h/2)|\alpha'|_B^2}\mathrm{e}^{\sqrt{-1}B(\alpha,\zeta)}\omega_\epsilon(U_\alpha^{B_0}).
\end{aligned}$$

By the same argument in Lemma 6.4, we have the assertion. □

Put (11) and Lemma 6.5 together, we obtain the following theorem.

Theorem 6.6. *Let A be a self-adjoint element in $\mathcal{H}_{1,d/2+2+\epsilon}$ ($\epsilon > 0$) and $E(B) = \|A(B)\|$ for $B \in \Omega$. Then there is a positive constant $C(\epsilon)$ such that*

$$|E(B_1) - E(B_2)| \leq C(\epsilon)\|A\|_{\mathcal{H}_{1,d/2+2+\epsilon}}\|B_1 - B_2\|,$$

where $\|B\|^2 = \mathrm{tr}_W(BB^*)$.

7. Final remark

One would wonder if there is actually a spectral gap for the Harper operator on a crystal lattice. There are some examples of crystal lattices which have gaps, at least for small magnetic flux. Because the band edges are continuous in magnetic flux, it is enough to find a crystal lattice whose transition operator (i.e. the magnetic flux $B = 0$ case) has gaps. Those examples are provided by Shirai [10] in the following way.

Let X be a regular graph of degree q. We construct a new regular graph $\mathcal{L}X$ from X. The vertices of $\mathcal{L}X$ are the unoriented edges of X and two vertices e_1 and e_2 of $\mathcal{L}X$ are adjacent when e_1 and e_2 are incidental as the edges of X. Thus the set $E(\mathcal{L}X)$ of all oriented edges of $\mathcal{L}X$ are given by

$$\{(e_1,e_2)|e_1,e_2 \in E, e_2 \neq \bar{e}_1, t(e_1) = o(e_2)\},$$

and the origin $o(e_1,e_2) = e_1$ and the terminus $t(e_1,e_2) = e_2$.

The $\mathcal{L}X$ is called the *line graph* of X (see Fig. 4). It is easy to see that $\mathcal{L}X$ is a regular graph of degree $2(q-1)$ and that $\mathcal{L}X$ is the Γ-covering graph of $\mathcal{L}X_0$ when X is the Γ-covering graph of X_0, so the line graph $\mathcal{L}X$ of a crystal lattice X is again a crystal lattice. Shirai computed the spectrum $\mathrm{Spec}(\Delta_{\mathcal{L}(X)})$ of the Laplacian $\Delta_{\mathcal{L}(X)}$ of $\mathcal{L}X$ to find

$$\mathrm{Spec}(\Delta_{\mathcal{L}(X)}) = \frac{q}{2(q-1)}\mathrm{Spec}(\Delta_{(X)}) \cup \left\{\frac{q}{q-1}\right\},$$

where $q/(q-1)$ is the eigenvalue of infinite multiplicity.

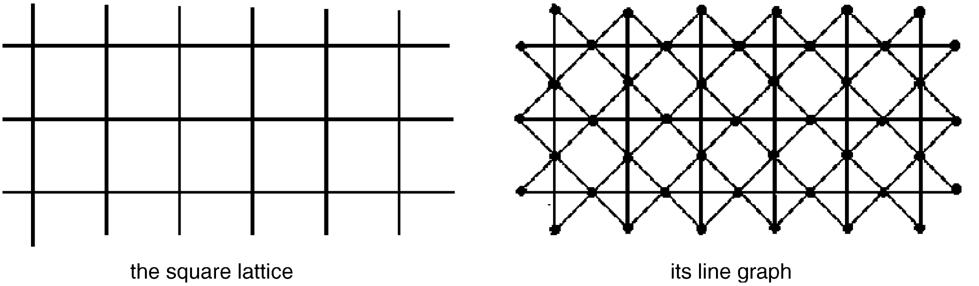

Fig. 4. The square lattice and its line graph.

Define the crystal lattice $\mathcal{L}^n X$ inductively by $\mathcal{L}^n X = \mathcal{L}(\mathcal{L}^{n-1} X)$. It is the regular graph of degree $2^n(q-2)+2$ and has

$$\mathrm{Spec}(\Delta_{\mathcal{L}^n(X)}) = \frac{q}{2^n(q-2)+2}\mathrm{Spec}(\Delta_X) \cup \bigcup_{k=0}^{n-1} \left\{ \frac{2(2^k(q-2)+2)}{2^n(q-2)+2} \right\}.$$

Note that $\mathrm{Spec}(\Delta_X) \subset [0,2]$ and that the right most edge of $q/(2^n(q-2)+2)\mathrm{Spec}(\Delta_X)$ coincides with the smallest eigenvalue $2q/(2^n(q-2)+2)$, when the right most edge of $\mathrm{Spec}(\Delta_X)$ is equal to 2. There are gaps between two eigenvalues next to each other. The transition operator L has gaps as $L = I - \Delta$.

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